Electrical Conductivity of Fermi Liquids. I.
Many-body Effect on the Drude Weight

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On the basis of the Fermi liquid theory, we investigate the many-body effect on the Drude weight. In a lattice system, the Drude weight D is modified by electron-electron interaction due to Umklapp processes, while it is not renormalized in a Galilean invariant system. This is explained by showing that the effective mass m' for $D \propto n/m'$ is defined through the current, not velocity, of quasiparticle. It is shown that the inequality $D > 0$ is required for the stability against the uniform shift of the Fermi surface. The result of perturbation theory applied for the Hubbard model indicates that D as a function of the density n is qualitatively modified around half filling $n \sim 1$ by Umklapp processes.

KEYWORDS: Fermi liquid theory, Drude weight, many-body effect, Umklapp process , metal-insulator transition,

§1. Introduction

In comparing theory and experiment of highly correlated electron systems, we must distinguish genuine many-body effects from one-body effects. This is trivial at least in a theoretical investigation based on a simple model. For example, in the Fermi liquid theory, the thermal mass measured in a specific heat experiment reflects the mass of quasiparticles, which can be heavily enhanced by interaction effects. On the other side, the effective mass m' defined in the Drude weight, which characterizes the metallic property of the system, does not generally coincide with the quasiparticle mass m^*. In fact it is shown that m' is not renormalized, or it is independent of electron-electron interactions in Galilean invariant systems. In a lattice system, Umklapp processes make the Drude weight depend on the electron-electron interaction and thus the Drude weight can be used as a direct probe of the metal-insulator transition.[1,2] This problem has been studied particularly in one dimensional systems,[2,3,4,5] where it is shown that the Drude weight D vanishes as the carrier density n approaches half filling, i.e., in the presence of the Coulomb interaction $U > 0$, we have

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$D \propto |1-n| = \delta$ for the low density limit $\delta \to 0$ of doped hole. Though the metal-insulator transition in this form might not be realized in general, we are interested in how this qualitative behavior of 'doped insulator' is reconciled with normal Fermi liquids in two or three dimensional systems, and we investigate and show how it is put in the context of the general theory of Fermi liquids when Umklapp processes are effective. In this respect, the effect of Umklapp processes in Fermi liquids was discussed for the cyclotron resonance frequency ω_c by Kanki and Yamada,[6] where it was shown that ω_c is not renormalized in an isotropic system. In a similar manner, we discuss that D is not renormalized in an isotropic system and then see in what manner D is modified in a lattice system.

In §2 we derive the expression for D on the basis of the Landau Fermi liquid theory and show that it is calculated as the second derivative of the energy E_p of the state $|p\rangle$, which is defined from the ground state by shifting the fermion distribution function n_k^0. The results obtained in §2 are derived microscopically in §3, where the Fermi liquid result of D is obtained using

$$D \equiv \lim_{\omega \to 0} \pi \omega \mathrm{Im} \sigma(\omega). \tag{1.1}$$

To supplement the general discussion of §§2 and 3, D is estimated explicitly in §4, where the effective mass for the Drude weight is calculated up to the second order in U of the Hubbard model in a square lattice. We observe that D as a function of n is modified qualitatively around half filling $n \sim 1$. The last section contains discussions. In this article, we investigate a system described by a single-band model of fermions at absolute zero. We are mainly interested in the collisionless region where the effect of a quasiparticle lifetime can be neglected. In Appendix we outline the derivation of one of the main results on the basis of the finite temperature formalism. A general theory at finite temperature will be presented in a subsequent paper.[7]

§2. Fermi Liquid Theory I

First we derive the uniform conductivity on the basis of the Landau theory of Fermi liquids.[8] In the presence of an applied electric field $\boldsymbol{E}$, the deviation δn_k from the ground-state distribution function n_k^0,

$$\delta n_k = n_k - n_k^0,$$

satisfies the Boltzmann equation,

$$(\boldsymbol{q} \cdot \boldsymbol{v}_k - \omega) \delta n_k + \boldsymbol{q} \cdot \boldsymbol{v}_k \delta(\mu - \varepsilon_k) \sum_{k'} f(k, k') \delta n_{k'}$$
$$+ \mathrm{i} e \boldsymbol{E} \cdot \boldsymbol{v}_k \delta(\mu - \varepsilon_k) = 0, \tag{2.1}$$

where ε_k is energy of the quasiparticle k and

$$v_k \equiv \frac{\partial \varepsilon_k}{\partial k}.$$

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For $\boldsymbol{q} = 0$ we obtain

$$\delta n_k = \frac{\mathrm{i}e\boldsymbol{E} \cdot \boldsymbol{v}_k \delta(\mu - \varepsilon_k)}{\omega}. \tag{2.2}$$

The total current induced by the field is

$$\boldsymbol{J} = \sum_k \delta n_k \boldsymbol{j}_k. \tag{2.3}$$

The conductivity tensor is defined by

$$\sigma_{\mu\nu}(\omega) = \frac{eJ_\mu}{\Omega E_\nu}, \tag{2.4}$$

where Ω is the total volume of the system. Hence we find

$$\sigma_{\mu\nu}(\omega) = \frac{\mathrm{i}e^2}{\omega} \left(\frac{n}{m'} \right)_{\mu\nu}, \tag{2.5}$$

where

$$\left(\frac{n}{m'} \right)_{\mu\nu} = \frac{1}{\Omega} \sum_k v_{k\mu} j_{k\nu} \delta(\mu - \varepsilon_k). \tag{2.6}$$

The Drude weight D, eq. (1.1), is given as the coefficient of the delta function $\delta(\omega)$ in $\mathrm{Re}\sigma_{\mu\nu}(\omega)$;

$$\mathrm{Re}\sigma_{\mu\nu}(\omega) = \pi e^2 \left(\frac{n}{m'} \right)_{\mu\nu} \delta(\omega). \tag{2.7}$$

Therefore, we obtain

$$D = \pi e^2 \left(\frac{n}{m'} \right)_{\mu\nu}.$$

The effective mass m', defined by eq. (2.6), plays an important role in our theory.

To see that electron-electron interaction does not modify the Drude weight in Galilean invariant systems, eq. (2.6) is transformed as follows,

$$\left(\frac{n}{m'} \right)_{\mu\nu} = \frac{1}{\Omega} \sum_k \left(-\frac{\partial n_k^0}{\partial k_\mu} \right) j_{k\nu} \tag{2.8}$$

$$= \frac{1}{\Omega} \sum_k n_k^0 \frac{\partial j_{k\nu}}{\partial k_\mu}. \tag{2.9}$$

Here we used

$$\delta(\mu - \varepsilon_k) = -\frac{\partial n_k^0}{\partial \varepsilon_k}, \tag{2.10}$$

i.e., the quasiparticle distribution function n_k^0 is represented by the step function $\theta(\mu - \varepsilon_k)$ at zero temperature. As a direct consequence of Galilean invariance, the relation $j_{k\nu} = k_\nu/m$, with the bare electron mass m, is not modified by electron-electron interactions. Therefore in this case we obtain

$$\left(\frac{n}{m'} \right)_{\mu\nu} = \frac{n}{m} \delta_{\mu\nu}, \tag{2.11}$$

where

$$n \equiv \frac{1}{\Omega} \sum_k n_k^0, \tag{2.12}$$

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and the Drude weight $D = \pi e^2 n/m$ is not affected by the interaction. If we assume $\partial j_{k\mu}/\partial k_\nu =$const. for simplicity, from eq. (2.9) we find

$$\left(\frac{1}{m'}\right)_{\mu\nu} = \frac{\partial j_{k\mu}}{\partial k_\nu} \neq \left(\frac{1}{m^*}\right)_{\mu\nu} \equiv \frac{\partial v_{k\mu}}{\partial k_\nu}, \tag{2.13}$$

where m^* represents the effective mass of the quasiparticle k. Hence it is concluded that the mass m' in the Drude weight is given through the current of quasiparticle, and that m', m^* and the bare (crystalline) mass m generally take different values. We remark that the effective mass m^* thus defined by $1/m^* \propto \langle v \rangle$ does not generally coincide with the thermal mass defined in the total density of states $m^*_{\text{th}} \propto \langle v^{-1} \rangle$, where the average is taken over the Fermi surface. The difference between m^* and m^*_{th} would be significant only in exceptional cases, as in the vicinity of the van Hove singularity where the former is less singular than the latter.

Now let us show that the right-hand side of eq. (2.6) is derived in terms of the state $|p\rangle$, which represents the state obtained by displacing the ground state configuration by p in the momentum space, i.e., $|p\rangle$ is derived from the ground state $|0\rangle$ by formally replacing the distribution function n_k^0 with n_{k-p}^0. By construction, the resulting state $|p\rangle$ carries a finite current $\boldsymbol{J_p}$.[9] Denoting the energy of the state $|p\rangle$ as E_p, we would like to derive $J_{p\mu} = \partial E_p/\partial p_\mu$ and express $(n/m')_{\mu\nu}$ as the second derivative of E_p with respect to p.

According to the Landau Fermi liquid theory, the current carried by the quasiparticle k is given by[8]

$$\boldsymbol{j}_k = \boldsymbol{v}_k + \sum_{k'} f(k,k')\boldsymbol{v}_{k'}\delta(\mu - \varepsilon_{k'}). \tag{2.14}$$

The spin dependence of the interaction function $f(k,k')$ was omitted for simplicity. Using $\delta n_k = n_{k-p}^0 - n_k^0 = -\boldsymbol{p} \cdot \boldsymbol{\nabla}_{\boldsymbol{k}} n^0 = \delta(\mu - \varepsilon_k)\boldsymbol{p} \cdot \boldsymbol{v}_k$, we obtain

$$\boldsymbol{J_p} = \sum_k \delta n_k \boldsymbol{j}_k = \sum_k \delta(\mu - \varepsilon_k)\boldsymbol{p} \cdot \boldsymbol{v}_k \boldsymbol{j}_k \tag{2.15}$$

$$= \sum_k \delta(\mu - \varepsilon_k)\boldsymbol{p} \cdot \boldsymbol{v}_k$$

$$\times \left(\boldsymbol{v}_k + \sum_{k'} f(k,k')\delta(\mu - \varepsilon_{k'})\boldsymbol{v}_{k'}\right). \tag{2.16}$$

Comparing eqs. (2.6) and (2.15), we obtain

$$\frac{\partial J_{p\mu}}{\partial p_\nu} = \Omega \left(\frac{n}{m'}\right)_{\mu\nu}. \tag{2.17}$$

On the other side, to estimate the change of the total energy

$$\delta E_p = \sum_k \varepsilon_k \delta n_k + \frac{1}{2}\sum_{k,k'} f(k,k')\delta n_k \delta n_{k'}, \tag{2.18}$$

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the deviation δn_k has to be expanded up to the second order in p: $\delta n_k = -\sum_i p_i \partial_{k_i} n^0 + \sum_{i,j} p_i p_j \partial_{k_i} \partial_{k_j} n^0/2$. Then we have

$$\frac{\partial E_p}{\partial p_\mu} = \sum_k \varepsilon_k \boldsymbol{p} \cdot \boldsymbol{\nabla}_{\boldsymbol{k}} \partial_{k_\mu} n^0$$
$$+ \sum_{k,k'} f(k,k') \delta(\mu - \varepsilon_k) \delta(\mu - \varepsilon_{k'}) \boldsymbol{p} \cdot \boldsymbol{v}_k v_{k'\mu},$$

(2.19)

where the symmetry relation $f(k,k') = f(k',k)$ was used in the second term. The first term is put into

$$-\sum_k \left(\partial_{k_\mu} \varepsilon_k\right) \boldsymbol{p} \cdot \boldsymbol{\nabla}_{\boldsymbol{k}} n^0 = \sum_k \delta(\mu - \varepsilon_k) \boldsymbol{p} \cdot \boldsymbol{v}_k v_{k\mu},$$

(2.20)

where we used $\sum_k \partial_{k_\mu} (\varepsilon_k \boldsymbol{p} \cdot \boldsymbol{\nabla}_{\boldsymbol{k}} n^0) = 0$. Hence we have

$$\frac{\partial E_p}{\partial p_\mu} = \sum_k \delta(\mu - \varepsilon_k) \boldsymbol{p} \cdot \boldsymbol{v}_k v_{k\mu}$$
$$+ \sum_{k,k'} f(k,k') \delta(\mu - \varepsilon_k) \delta(\mu - \varepsilon_{k'}) \boldsymbol{p} \cdot \boldsymbol{v}_k v_{k'\mu}.$$

(2.21)

As a result, from eqs. (2.16) and (2.21) we obtain

$$J_{p\mu} = \frac{\partial E_p}{\partial p_\mu},$$

(2.22)

and using eq. (2.17) we conclude

$$\frac{\partial J_{p\mu}}{\partial p_\nu} = \frac{\partial^2 E_p}{\partial p_\mu \partial p_\nu} = \Omega \left(\frac{n}{m'}\right)_{\mu\nu}.$$

(2.23)

As the assumption of an isotropic system, to regard $f(k,k')$ just as a function of $\cos\theta = \boldsymbol{k} \cdot \boldsymbol{k}'/kk'$, is not used in the above derivation, the result (2.23) is generally applied even for an anisotropic system.[10]

§3. Fermi Liquid Theory II

In order to describe the state $|p\rangle$ microscopically, we introduce the Green's function

$$G_p'^0(k) = \frac{1}{\omega - \varepsilon_k + \mu + \mathrm{i}\,\mathrm{sign}(\varepsilon_{k-p} - \mu)}.$$

(3.1)

We use the abbreviated notation such as $k = (\boldsymbol{k}, \omega)$ and $k' = (\boldsymbol{k}', \omega')$. In eq. (3.1), ε_k represents energy of the bare particle k. In the following, the energy of the quasiparticle k will be denoted by ε_k^*. To calculate E_p of the state $|p\rangle$, the Green's function (3.1) is to be used instead of

$$G^0(k) = \frac{1}{\omega - \varepsilon_k + \mu + \mathrm{i}\,\mathrm{sign}(\varepsilon_k - \mu)},$$

which is usually used to describe the ground state.[11] In place of eq. (3.1), we may equivalently use

$$G_p^0(k) = \frac{1}{\omega - \varepsilon_{k+p} + \mu + \mathrm{i}\,\mathrm{sign}(\varepsilon_k - \mu)}.$$

(3.2)

This Green's function describes the system which has the same Fermi surface as for the ground state, but with the shifted one-body energy ε_{k+p} instead of ε_k. For example, the one-body part of the total energy of $|p\rangle$ is expressed in terms of $G_p'^0(k) = G_p^0(k - p)$,

$$\begin{aligned}
E_p^0 &= \sum_k \varepsilon_k \langle \hat{c}_k^\dagger \hat{c}_k \rangle_p^0 = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \varepsilon_k G_p'^0(k) \\
&= -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \varepsilon_{k+p} G_p^0(k) \\
&= -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \varepsilon_{k+p} G^0(k) \\
&= \sum_k \varepsilon_{k+p} \langle \hat{c}_k^\dagger \hat{c}_k \rangle^0.
\end{aligned}$$

(3.3)

To evaluate $\partial E_p / \partial p$, the p-dependence in the argument of $G_p^0(k - p)$ can be neglected by virtue of quasimomentum conservation at each interaction point, or we can replace $G_p'^0(k)$ in E_p by $G_p^0(k)$ without loss of generality. Physically, this is related to the fact that replacing the distribution function n_k by n_{k-p} in the expression for E_p is equivalent to replacing ε_k by ε_{k+p}. This point will be seen below in §4 for a simple case. As a result, for the energy correction $\Delta E_p \equiv E_p - E_p^0$ caused by electron-electron interaction, we obtain the equation

$$\frac{\partial \Delta E_p}{\partial p_\mu} = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\partial G_p^0(k)}{\partial p_\mu} \Sigma_p'(k),$$

(3.4)

where $\Sigma_p'(k)$ is the total self energy including improper as well as proper parts. In terms of the identity

$$\frac{\partial G_p^0(k)}{\partial p_\mu} = v_{k+p\mu} G_p^0(k)^2,$$

(3.5)

we have

$$\frac{\partial \Delta E_p}{\partial p_\mu} = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} v_{k+p\mu} G_p^0(k)^2 \Sigma_p'(k)$$

(3.6)

$$= -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} v_{k+p\mu} G_p^0(k) \Sigma_p(k) G_p(k),$$

(3.7)

where we used the relation $\Sigma_p'(k) G_p^0(k) = \Sigma_p(k) G_p(k)$ for the proper self-energy $\Sigma_p(k)$ and the dressed Green's function defined by

$$G_p(k)^{-1} = G_p^0(k)^{-1} - \Sigma_p(k).$$

(3.8)

Thus, together with

$$\frac{\partial E_p^0}{\partial p_\mu} = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} v_{k+p\mu} G_p^0(k),$$

(3.9)

eq. (3.7) leads to

$$\frac{\partial E_p}{\partial p_\mu} = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} v_{k+p\mu} G_p(k),$$

(3.10)

where we used

$$G_p(k) = G_p^0(k) + G_p^0(k)\Sigma_p(k)G_p(k).$$

(3.11)

The total current of the state $|p\rangle$ is just given by eq. (3.10), since we find

$$J_p \equiv \sum_k v_k \langle \hat{c}_k^\dagger \hat{c}_k \rangle_p = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} v_k G_p'(k)$$

$$= -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} v_{k+p} G_p(k).$$

(3.12)

Therefore we obtain

$$J_{p\mu} = \frac{\partial E_p}{\partial p_\mu}.$$

(3.13)

Now we derive the other results of the previous section on the basis of the microscopic Fermi liquid theory using well-known Fermi liquid relations.[12, 13] To calculate a linear response to the vector potential $\boldsymbol{A}$, the total current operator is defined by

$$\hat{J}_\mu = \hat{v}_\mu - e \sum_\nu \hat{\varepsilon}''_{\mu\nu} A_\nu,$$

(3.14)

where

$$\hat{v}_\mu \equiv \sum_k v_{k\mu} \hat{c}_k^\dagger \hat{c}_k, \qquad v_{k\mu} = \frac{\partial \varepsilon_k}{\partial k_\mu},$$

(3.15)

and

$$\hat{\varepsilon}''_{\mu\nu} \equiv \sum_k \frac{\partial^2 \varepsilon_k}{\partial k_\mu \partial k_\nu} \hat{c}_k^\dagger \hat{c}_k.$$

(3.16)

Then the dynamical conductivity is given by

$$\sigma_{\mu\nu}(\omega) = \frac{e^2}{\omega + \mathrm{i}0} \left(\mathrm{i} K_{\mu\nu}(\omega + \mathrm{i}0) + \frac{1}{\Omega} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\partial^2 \varepsilon_k}{\partial k_\mu \partial k_\nu} G(k) e^{+\mathrm{i}\omega 0} \right),$$

(3.17)

where $K_{\mu\nu}(\omega)$, defined by

$$K_{\mu\nu}(\omega) = -\frac{\mathrm{i}}{\Omega} \int_{-\infty}^{\infty} \langle 0 | \mathrm{T} \hat{v}_\mu(t) \hat{v}_\nu | 0 \rangle \exp(\mathrm{i}\omega t) \mathrm{d}t,$$

(3.18)

corresponds to the uniform limit $k \to 0$ of the current correlation function $K_{\mu\nu}(k)$. For the Green's function $G(k)$ of the ground state, we omit the subscript $p \, (= 0)$ for simplicity. In terms of the vertex function $\Lambda_\nu(k'; k)$, $K_{\mu\nu}(k)$ is given by

$$K_{\mu\nu}(k) = -\frac{\mathrm{i}}{\Omega} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} v_{k\mu} G(k' + k/2) G(k' - k/2) \Lambda_\nu(k'; k).$$

(3.19)

The vertex function satisfies the following equation;

$$\Lambda_\mu(k'; k) = v_{k'\mu} - \mathrm{i} \int \frac{\mathrm{d}^4 k''}{(2\pi)^4} \Gamma^{(0)}(k', k'') G(k'' + k/2) G(k'' - k/2) \Lambda_\mu(k''; k),$$

(3.20)

where $\Gamma^{(0)}(k, k')$ is the irreducible four-point vertex function. As is well known, in the limit $k = (\boldsymbol{k}, \omega) \to 0$, which corresponds to the k-ω region in the vicinity of the Fermi surface, the product of the Green's functions $G(k' + k/2)G(k' - k/2)$ does not behave regularly but it is written

$$G(k' + k/2)G(k' - k/2) = G(k')^2 + 2\pi i z_{k'}^2 \frac{\boldsymbol{k} \cdot \boldsymbol{v}_{k'}^*}{\omega - \boldsymbol{k} \cdot \boldsymbol{v}_{k'}^*} \delta(\mu - \varepsilon_{k'}^*)\delta(\omega'). \tag{3.21}$$

The first term $G(k')^2$ represents a regular part of the left-hand side. In particular, we are interested in the ω-limit ($k/\omega = 0$) and the k-limit ($k/\omega = \infty$) of eq. (3.21), for which we obtain the relation

$$\left\{ G(k')^2 \right\}^\omega - \left\{ G(k')^2 \right\}^k = 2\pi i z_{k'}^2 \delta(\mu - \varepsilon_{k'}^*)\delta(\omega'). \tag{3.22}$$

In eq. (3.21), v_k^* represents the velocity of the quasiparticle k,

$$v_k^* = \frac{\partial \varepsilon_k^*}{\partial k}. \tag{3.23}$$

The quasiparticle energy ε_k^* is given as a pole of the Green's function,

$$G^{-1}(k, \varepsilon_k^* - \mu) = \varepsilon_k^* - \varepsilon_k - \Sigma(k, \varepsilon_k^* - \mu) = 0, \tag{3.24}$$

where the many-body effect in the one-particle spectrum is embodied in the selfenergy $\Sigma(k, \omega)$. The renormalization factor z_k is defined by

$$z_k^{-1} = 1 - \left.\frac{\partial \Sigma(k, \omega)}{\partial \omega}\right|_{\omega=0}. \tag{3.25}$$

Differentiating eq. (3.24) with respect to k, we have

$$v_k^* \left(1 - \left.\frac{\partial \Sigma(k, \omega)}{\partial \omega}\right|_{\omega=0} \right) - v_k - \left.\frac{\partial \Sigma(k, \omega)}{\partial k}\right|_{\omega=0} = 0, \tag{3.26}$$

thus we find

$$v_k^* = z_k \left(v_k + \frac{\partial \Sigma(k, 0)}{\partial k} \right). \tag{3.27}$$

Next we consider the derivative of the selfenergy,

$$\frac{\partial \Sigma(k, \omega)}{\partial k_\mu} = -i \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \Gamma^{(0)}(k, k') \frac{\partial}{\partial k_\mu'} G(k', \omega'), \tag{3.28}$$

for which, using

$$\frac{\partial}{\partial k_\mu'} G(k', \omega') = \left\{ G(k')^2 \right\}^k \left(v_{k'\mu} + \frac{\partial \Sigma(k', \omega')}{\partial k_\mu'} \right), \tag{3.29}$$

we obtain

$$\frac{\partial \Sigma(k, \omega)}{\partial k_\mu} = -i \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \Gamma^{(0)}(k, k') \left\{ G(k')^2 \right\}^k \left(v_{k'} + \frac{\partial \Sigma(k', \omega')}{\partial k'} \right). \tag{3.30}$$

Note that the k-limit $\left\{ G(k')^2 \right\}^k$ appears in eq. (3.29) when the derivative is taken with respect to k' for fixed ω'. Comparing eq. (3.30) with

$$\Lambda_\mu^k(k') = v_{k'\mu} - i \int \frac{\mathrm{d}^4 k''}{(2\pi)^4} \Gamma^{(0)}(k', k'') \left\{ G(k'')^2 \right\}^k \Lambda_\mu^k(k''), \tag{3.31}$$

which is derived from eq. (3.20), we find

$$\Lambda_\mu^k(k') = v_{k'\mu} + \frac{\partial \Sigma(k', \omega')}{\partial k'_\mu}, \tag{3.32}$$

hence,

$$v_{k'\mu}^* = z_{k'} \Lambda_\mu^k(k'), \tag{3.33}$$

because of eq. (3.27).

Using the above results, we can write the k-limit of $K_{\mu\nu}(k)$, eq. (3.19), as

$$K_{\mu\nu}^k = -\frac{\mathrm{i}}{\Omega} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} v_{k'\mu} \left\{ G(k')^2 \right\}^k \Lambda_\nu^k(k') \tag{3.34}$$

$$= -\frac{\mathrm{i}}{\Omega} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} v_{k'\mu} \frac{\partial}{\partial k'_\nu} G(k') \tag{3.35}$$

$$= \frac{\mathrm{i}}{\Omega} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{\partial v_{k'\mu}}{\partial k'_\nu} G(k'), \tag{3.36}$$

where we used eqs. (3.29) and (3.32) in the second line. Substituting eq. (3.36) for the second term in the parenthesis of eq. (3.17), we can write the conductivity as

$$\sigma_{\mu\nu}(\omega) = \frac{\mathrm{i}e^2}{\omega + \mathrm{i}0} \left(K_{\mu\nu}(\omega + \mathrm{i}0) - K_{\mu\nu}^k \right). \tag{3.37}$$

Thus, the effective mass m' in the Drude weight,

$$\mathrm{Re}\,\sigma_{\mu\nu}(\omega) = \pi e^2 \left(\frac{n}{m'} \right)_{\mu\nu} \delta(\omega) + \sigma_{\mathrm{inc}}, \tag{3.38}$$

is given by the difference between the ω-limit and the k-limit of $K_{\mu\nu}(k)$;

$$\left(\frac{n}{m'} \right)_{\mu\nu} = K_{\mu\nu}^\omega - K_{\mu\nu}^k. \tag{3.39}$$

In other words, the Drude weight is determined by the singular (coherent) part of $G^2(k')$, eq. (3.22). To estimate eq. (3.39), from eq. (3.20) we obtain

$$\Lambda_\mu^\omega(k) = \Lambda_\mu^k(k) - \mathrm{i} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \Gamma^\omega(k, k') \left(\left\{ G(k')^2 \right\}^\omega - \left\{ G(k')^2 \right\}^k \right) \Lambda_\mu^k(k'). \tag{3.40}$$

Thus, eqs. (3.22), (3.33), (3.40) lead to

$$z_k \Lambda_\mu^\omega(k) = v_{k\mu}^* + \sum_{k'} f(k, k') v_{k'\mu}^* \delta(\mu - \varepsilon_{k'}^*) \equiv j_{k\mu}^*, \tag{3.41}$$

where

$$f(k, k') \equiv z_k z_{k'} \Gamma^\omega(k, k'). \tag{3.42}$$

Since we have

$$K_{\mu\nu}^\omega = K_{\mu\nu}^k + \frac{1}{\Omega} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \Lambda_\mu^k(k') \left(\left\{ G(k')^2 \right\}^\omega - \left\{ G(k')^2 \right\}^k \right) \Lambda_\mu^\omega(k'), \tag{3.43}$$

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which is derived from eq. (3.19), using eqs. (3.33), (3.22) and (3.41), for eq. (3.39) we finally obtain

$$\left(\frac{n}{m'}\right)_{\mu\nu} = \frac{1}{\Omega}\sum_k v_{k\mu}^* j_{k\nu}^* \delta(\mu - \varepsilon_k^*), \tag{3.44}$$

in agreement with eq. (2.6). Hence, following the argument given below eq. (2.6) and using $j_k^* = v_k = k/m$, we can conclude the absence of renormalization in the Drude weight in Galilean invariant systems. In particular, when $\partial v_{k\mu}/\partial k_\nu = \delta_{\mu\nu}/m$, eq. (3.36) gives

$$K_{\mu\nu}^k = \frac{\delta_{\mu\nu}}{m}\frac{\mathrm{i}}{\Omega}\int\frac{\mathrm{d}^4 k'}{(2\pi)^4}G(k') = -\frac{n}{m}\delta_{\mu\nu}. \tag{3.45}$$

Therefore, in this case, the sum rule,

$$\int_{-\infty}^\infty \mathrm{Re}\sigma_{\mu\nu}(\omega)\mathrm{d}\omega = \frac{\pi e^2}{\Omega}\left\langle\hat{\varepsilon}_{\mu\nu}''\right\rangle \tag{3.46}$$

$$= \frac{\pi n e^2}{m}\delta_{\mu\nu}, \tag{3.47}$$

is saturated with the Drude weight, hence $K_{\mu\nu}^\omega = 0$. In general, a lost weight in the coherent part is transfered to an incoherent part;

$$\left(\frac{n}{m}\right)_{\mu\nu} - \left(\frac{n}{m'}\right)_{\mu\nu} = -\frac{1}{\pi}\int_{-\infty}^\infty\frac{\mathrm{Im}K_{\mu\nu}(\omega')}{\omega'}\mathrm{d}\omega' \geq 0, \tag{3.48}$$

where we defined the effective mass in the total weight,

$$\left(\frac{n}{m}\right)_{\mu\nu} \equiv \frac{1}{\Omega}\left\langle\hat{\varepsilon}_{\mu\nu}''\right\rangle. \tag{3.49}$$

The incoherent part of the spectrum cannot be described in the framework of the Fermi liquid theory.[14]

In the same manner as we did for $G(k)$, we can derive the Ward identity for the Green's function $G_p(k)$ defined in eq. (3.8). Let us introduce ε_{kp}^* defined by

$$\varepsilon_{kp}^* - \varepsilon_{k+p} - \Sigma_p(k, \varepsilon_{kp}^* - \mu) = 0. \tag{3.50}$$

Differentiating this with respect to p, we find

$$\left.\frac{\partial\varepsilon_{kp}^*}{\partial p}\right|_{p=0} = z_k\left(v_k + \left.\frac{\partial}{\partial p}\Sigma_p(k,0)\right|_{p=0}\right). \tag{3.51}$$

Similarly as in eqs. (3.28) - (3.33), using

$$\frac{\partial\Sigma_p(k',\omega')}{\partial p_\mu} = -\mathrm{i}\int\frac{\mathrm{d}^4 k''}{(2\pi)^4}\Gamma^{(0)}(k',k'')\frac{\partial}{\partial p_\mu}G_p(k'',\omega''), \tag{3.52}$$

and

$$\frac{\partial}{\partial p_\mu}G_p(k',\omega') = \left\{G(k')^2\right\}^\omega\left(v_{k'\mu} + \frac{\partial\Sigma_p(k',\omega')}{\partial p_\mu}\right), \tag{3.53}$$

we get

$$\frac{\partial \Sigma_p(k', \omega')}{\partial p_\mu} = -\mathrm{i} \int \frac{\mathrm{d}^4 k''}{(2\pi)^4} \Gamma^{(0)}(k', k'') \left\{ G(k'')^2 \right\}^\omega \left(v_{k''\mu} + \frac{\partial \Sigma_p(k'', \omega'')}{\partial p_\mu} \right). \tag{3.54}$$

Here the regular function $\{G(k'')^2\}^\omega$ in the ω-limit should be used. This is because the pole ε_{kp}^* of $G_p(k, \omega)$ adiabatically moves to ε_k^* as a function of p without crossing the Fermi surface, since the Fermi surface for $G_p(k')$ is held fixed as a function of p. Thus, because of

$$\Lambda_\mu^\omega(k') = v_{k'\mu} - \mathrm{i} \int \frac{\mathrm{d}^4 k''}{(2\pi)^4} \Gamma^{(0)}(k', k'') \left\{ G(k'')^2 \right\}^\omega \Lambda_\mu^\omega(k''), \tag{3.55}$$

which is obtained from eq. (3.20), we find

$$\Lambda_\mu^\omega(k') = v_{k'\mu} + \frac{\partial \Sigma_p(k', \omega')}{\partial p_\mu}, \tag{3.56}$$

and eq. (3.51) gives

$$\left. \frac{\partial \varepsilon_{kp}^*}{\partial p} \right|_{p=0} = j_{k\mu}^*, \tag{3.57}$$

where we used eqs. (3.41) and (3.51). Equation (3.57) is to be compared with eq. (3.23). In particular, under the assumption

$$v_k + v_{k'} - v_{k+q} - v_{k'-q} = 0, \tag{3.58}$$

we have

$$v_k \frac{\partial \Sigma_p(k, \omega)}{\partial \omega} + \frac{\partial \Sigma_p(k, \omega)}{\partial p} = 0. \tag{3.59}$$

It is easy to check this equation for a few diagrams of lower order. Therefore, in this case, we see from eqs. (3.51) and (3.57) that the quasiparticle current is not renormalized; $j_k^* = v_k$. This equation led us to conclude the absence of renormalization in D.

To derive eq. (2.23), we write the ω-limit of $K_{\mu\nu}(k)$ as

$$K_{\mu\nu}^\omega = -\frac{\mathrm{i}}{\Omega} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} v_{k'\mu} \left\{ G(k')^2 \right\}^\omega \Lambda_\nu^\omega(k')$$

$$= -\frac{\mathrm{i}}{\Omega} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} v_{k'\mu} \frac{\partial}{\partial p_\nu} G_p(k'), \tag{3.60}$$

which corresponds to eq. (3.35) for $K_{\mu\nu}^\infty$. Subtracting eq. (3.36) from eq. (3.60),

$$K_{\mu\nu}^\omega - K_{\mu\nu}^k = -\frac{\mathrm{i}}{\Omega} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \left(v_{k'\mu} \frac{\partial}{\partial p_\nu} G_p(k') + \frac{\partial v_{k'\mu}}{\partial k_\nu'} G(k') \right)$$

$$= -\frac{\mathrm{i}}{\Omega} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{\partial}{\partial p_\nu} \left(v_{k'+p\mu} G_p(k') \right) \bigg|_{p=0}.$$

Therefore, in terms of eqs. (3.12) and (3.13), for eq. (3.39) we finally obtain

$$\left(\frac{n}{m'} \right)_{\mu\nu} = \frac{1}{\Omega} \frac{\partial J_{p\mu}}{\partial p_\nu} = \frac{1}{\Omega} \frac{\partial^2 E_p}{\partial p_\mu \partial p_\nu}. \tag{3.61}$$

§4. Perturbation Theory

4.1 Linear response theory

The effective mass in the Drude weight, defined in eq. (2.6) or eq. (3.44), which is derived on the basis of the Fermi liquid theory, gives us a definite picture on how the Drude weight is renormalized by the many-body effect. However, to estimate the effective mass perturbatively, the formula (3.61) is practically useful. Not only to verify this formula but to see how Umklapp processes affect the Drude weight, we apply a finite-order perturbation theory in the following.

In the spectral representation, the dynamical conductivity (3.17) is written as

$$\sigma_{\mu\nu}(\omega) = \frac{e^2}{i\omega}\left(\frac{i}{\Omega}\int_{-\infty}^{\infty}\langle 0|\mathrm{T}\hat{v}_\mu(t)\hat{v}_\nu|0\rangle\exp(i\omega t)\mathrm{d}t - \frac{1}{\Omega}\langle 0|\hat{\varepsilon}''_{\mu\nu}|0\rangle\right). \tag{4.1}$$

For $\hat{v}_\mu$ and $\hat{\varepsilon}''_{\mu\nu}$ in this expression, we use eqs. (3.15) and (3.16), where the sum over the spin component σ is implicitly assumed in the summation Σ_k. Then, taking the limit $\omega \to 0$, we get

$$\left(\frac{n}{m'}\right)_{\mu\nu} \equiv -\frac{2}{\Omega}\sum_{r\neq 0}\frac{\langle 0|\hat{v}_\mu|r\rangle\langle r|\hat{v}_\nu|0\rangle}{E_r - E_0} + \frac{1}{\Omega}\langle 0|\hat{\varepsilon}''_{\mu\nu}|0\rangle. \tag{4.2}$$

In this expression, the sum is taken over the exact excited states $|r\rangle$ of the system. The right-hand side of eq. (4.2) is derived also from the energy shift $E_p - E_0$ caused by the perturbation,

$$H_p - H \equiv \sum_{k\sigma}(\varepsilon_{k+p} - \varepsilon_k)\hat{c}^\dagger_{k\sigma}\hat{c}_{k\sigma}$$

$$\simeq \boldsymbol{p}\cdot\hat{\boldsymbol{v}} + \frac{1}{2}\sum_{\mu,\nu}p_\mu p_\nu \hat{\varepsilon}''_{\mu\nu}. \tag{4.3}$$

As a result we obtain the relation

$$\frac{\partial^2 E_p}{\partial p_\mu \partial p_\nu} = \Omega\left(\frac{n}{m'}\right)_{\mu\nu}, \tag{4.4}$$

for

$$E_p = \langle p|H|p\rangle = \langle 0|H_p|0\rangle, \tag{4.5}$$

where $|p\rangle$ is the ground state of H_p.

At this point, it is clear that the method given here has a close connection with Kohn's discussion[1,2] that the effective mass defined in eq. (2.7) is derived by investigating variation of the ground state energy under changes in the boundary condition of a *finite* system. In a finite system the change of the boundary condition causes a uniform shift of discrete momentum quantum numbers. The point is that, in our theory, the chemical-potential shift eq. (4.3) should be treated as a perturbation. Since the perturbation adiabatically applied to the system cannot deform the Fermi surface, we cannot reach $|p\rangle$ from $|0\rangle$ by perturbation theory, but we are led to the excited state of the energy E_p, eq. (4.5), instead of the ground state of H_p. In practice, if E_0 ($= \langle 0|H|0\rangle$) is expressed in terms of bare quantities and interaction parameters, E_p is easily obtained from E_0 by

replacing the distribution function n_k with n_{k-p}. Thus, as a method to calculate the Drude weight, our theory is equivalent to Kohn's argument, so that eq. (3.61) is generally applied with the proviso that E_p is regarded as the energy of the state $|p\rangle$ derived from $|0\rangle$. The investigation made in the preceding sections is still instructive, since it clarifies the physical meaning of the Drude weight D: We must assume $D > 0$ for the Fermi liquid state to be stable against the shift of the Fermi surface in the momentum space.

4.2 Drude weight

To evaluate eq. (4.2) by perturbation theory, the ground state $|0\rangle$ is expanded up to terms of the second order in the interaction $V' \equiv V - \langle V \rangle$,[15)]

$$|0\rangle = |0\rangle_0 + |0\rangle_1 + |0\rangle_2,$$

$$|0\rangle_1 = -\sum_{r \neq 0} \frac{|r\rangle \langle r|V'|0\rangle_0}{E_r - E_0}, \tag{4.6}$$

$$|0\rangle_2 = \sum_{r \neq 0, s \neq 0} \frac{|r\rangle \langle r|V'|s\rangle \langle s|V'|0\rangle_0}{(E_r - E_0)(E_s - E_0)}$$

$$- \frac{|0\rangle_0}{2} \sum_{r \neq 0} \frac{|\langle r|V'|0\rangle_0|^2}{(E_r - E_0)^2}. \tag{4.7}$$

To put it concretely, let us study the Hubbard model

$$H = T + V = \sum_{k,\sigma} \varepsilon_k \hat{c}_{k\sigma}^\dagger \hat{c}_{k\sigma} + \sum_i U \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}. \tag{4.8}$$

We assume that the effective mass in eq. (4.2) becomes isotropic in the x-y plane, $m'_{\mu\nu} \equiv m' \delta_{\mu\nu}$. In the zeroth order in U, the second term of eq. (4.2) gives

$$\left(\frac{n}{m'} \right)_0 = {}_0\langle 0| \hat{\varepsilon}''_{\mu\mu} |0\rangle_0 \tag{4.9}$$

$$= \frac{1}{\Omega} \sum_{k\sigma} n_k \frac{\partial^2 \varepsilon_k}{\partial k_\mu^2} \tag{4.10}$$

$$= \frac{1}{\Omega} \frac{\partial^2}{\partial p^2} E_p^{(0)} \bigg|_{p=0}, \tag{4.11}$$

where

$$n_k = {}_0\langle 0| \hat{c}_{k\sigma}^\dagger \hat{c}_{k\sigma} |0\rangle_0, \tag{4.12}$$

and

$$E_p^{(0)} = \sum_{k\sigma} n_k \varepsilon_{k+p} = \sum_{k\sigma} n_{k-p} \varepsilon_k, \tag{4.13}$$

or, similarly we have

$$\left(\frac{n}{m'} \right)_0 = -\frac{1}{\Omega} \sum_{k\sigma} \frac{\partial n_k}{\partial k_\mu} \frac{\partial \varepsilon_k}{\partial k_\mu} = -\frac{1}{\Omega} \sum_{k\sigma} \frac{\partial n_k}{\partial \varepsilon_k} v_{k\mu}^2. \tag{4.14}$$

It is noted in eq. (4.13) that shifting the distribution $n_k \to n_{k-p}$ is formally equivalent to replacing ε_{k+p} by ε_k for fixed n_k. Accordingly, in eq. (4.10), $(n/m')_0$ is expressed as an average of $\partial^2 \varepsilon_k / \partial k_\mu^2$ over the states *below* the Fermi level, while in eq. (4.14) it is given by an average of $v_{k\mu}^2$ over the states *at* the Fermi level, as in eq. (3.44). To estimate $1/m_0'$ numerically, eq. (4.10) is more suitable than eq. (4.14).

For the correction $(n/m')_2$ of order $O(U^2)$, from the first term of eq. (4.2) we obtain

$$-\frac{2}{\Omega} \sum_{r \neq 0} \frac{|\langle r|\hat{v}_x|0\rangle_1|^2}{E_r - E_0} = -2\frac{U^2}{\Omega^3} \sum_{k_1 \sim k_4} \frac{(1 - n_{k_1})(1 - n_{k_2})n_{k_3}n_{k_4}}{(\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{k_3} - \varepsilon_{k_4})^3}(v_{k_1 x} + v_{k_2 x} - v_{k_3 x} - v_{k_4 x})^2 \delta_{k_1 + k_2 - k_3 - k_4},$$

$$(4.15)$$

where

$$\delta_q \equiv \frac{1}{\Omega} \sum_i e^{-iqr_i}, \tag{4.16}$$

and from the second term of eq. (4.2),

$$\frac{1}{\Omega}\Big({}_1\langle 0|\hat{\varepsilon}_{\mu\mu}''|0\rangle_1 + {}_0\langle 0|\hat{\varepsilon}_{\mu\mu}''|0\rangle_2 + {}_2\langle 0|\hat{\varepsilon}_{\mu\mu}''|0\rangle_0\Big)$$

$$= \frac{U^2}{\Omega^3} \sum_{k_1 \sim k_4} \frac{(1 - n_{k_1})(1 - n_{k_2})n_{k_3}n_{k_4}}{(\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{k_3} - \varepsilon_{k_4})^2}\left(\frac{\partial^2 \varepsilon_{k_1}}{\partial k_x^2} + \frac{\partial^2 \varepsilon_{k_2}}{\partial k_x^2} - \frac{\partial^2 \varepsilon_{k_3}}{\partial k_x^2} - \frac{\partial^2 \varepsilon_{k_4}}{\partial k_x^2}\right)\delta_{k_1 + k_2 - k_3 - k_4} \tag{4.17}$$

As a result,

$$\left(\frac{n}{m'}\right)_2 = \text{eq. (4.15)} + \text{eq. (4.17)}, \tag{4.18}$$

and we find

$$\left(\frac{n}{m'}\right)_2 = \frac{1}{\Omega}\frac{\partial^2}{\partial p^2}E_p^{(2)}\bigg|_{p=0}, \tag{4.19}$$

where

$$E_p^{(2)} = -\frac{U^2}{\Omega^2} \sum_{k_1 \sim k_4} \frac{(1 - n_{k_1})(1 - n_{k_2})n_{k_3}n_{k_4}}{\varepsilon_{k_1 + p} + \varepsilon_{k_2 + p} - \varepsilon_{k_3 + p} - \varepsilon_{k_4 + p}}\delta_{k_1 + k_2 - k_3 - k_4} \tag{4.20}$$

$$= -\frac{U^2}{\Omega^2} \sum_{k_1 \sim k_4} \frac{(1 - n_{k_1 - p})(1 - n_{k_2 - p})n_{k_3 - p}n_{k_4 - p}}{\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{k_3} - \varepsilon_{k_4}}\delta_{k_1 + k_2 - k_3 - k_4}. \tag{4.21}$$

The above results, following eqs. (4.11) and (4.19), verify the relation (3.61) up to terms of the second order in U. In particular, eqs. (4.15) and (4.17) indicate $(n/m')_2 = 0$ for

$$v_k + v_{k'} - v_{k+q} - v_{k'-q} = 0, \tag{4.22}$$

i.e., the violation of the group-velocity conservation leads to renormalization of the Drude weight, in accordance with the note made below eq. (3.58). Recently this point was discussed by Maebashi and Fukuyama.[16]

In Fig. 1, the ratio n/m' calculated with eqs. (4.11) and (4.19) are displayed as a function of n for $U = 0$ and $U = 1$ for the square lattice with $4t = 1$ and also for the case where the next-nearest-neighbor hopping $t' = -0.2t$ is included. In the figure, n/m for the total weight defined in
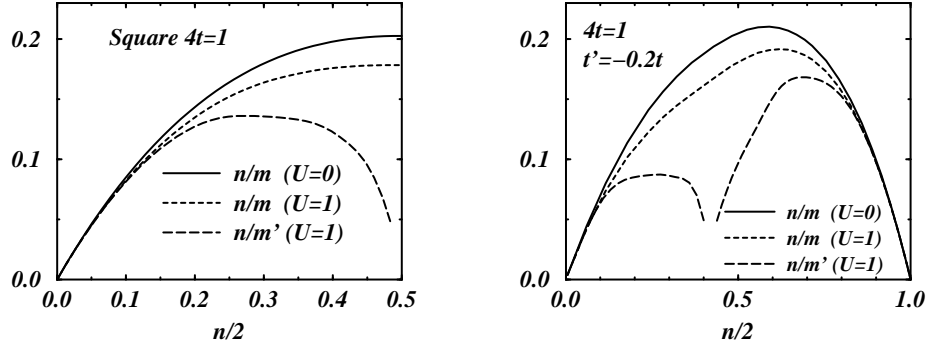
Fig. 1. (a) n/m' for $U = 0$, $U = 1$ and n/m for $U = 1$ are displayed as a function of $n/2$ for the Hubbard model in the square lattice with $4t = 1$ (left). (b) The same quantities in the presence of the next-nearest-neighbor hopping $t' = -0.2t$ (right). The Drude weight D and the total weight of $\sigma(\omega)$ are proportional to n/m' and n/m, respectively. The curves for n/m' ($U = 1$) are not shown around the van Hove singularity.

eq. (3.49), which is given by eqs. (4.14) and (4.17), is also shown. It is clear from the figure that the many-body effect modifies the Drude weight,

$$D = \frac{\pi e^2 n}{m'}.$$

For $U = 1$, we see n/m' decreases precipitously as n approaches the van Hove singularity. In the second-order correction $-(n/m')_2$, we found that the part $-(n/m')_2^{\mathrm{n}}$ that is not due to Umklapp processes varies monotonically. For example, in Fig. 1 (a), $-(n/m')_2^{\mathrm{n}}$ increases almost linearly up to less than 0.03 as n approaches half filling. The small and non-singular behavior of $-(n/m')_2^{\mathrm{n}}$ due to normal processes is understood by observing that the relevant normal processes are not restricted in a narrow region around the Fermi level, where the effect of the van Hove singularity is prominent: For the Fermi surfaces of our example, energy conservation excludes normal processes at the Fermi level.[16] Thus in our example of a single-band model, the renormalization of the Drude weight and the conspicuous behavior shown in the figure are primarily caused by Umklapp processes, that is, the renormalization due to the normal processes violating the velocity conservation[16] are qualitatively unimportant here.

We have $dD/dn = \pi e^2/m' > 0$ for the Drude weight in an isotropic system, since $m' (\equiv m)$ in this case is independent of n. Correspondingly, even in a lattice system, it is convenient to regard current-carrying carriers as particles (holes) when dD/dn is positive (negative). For instance, in the $U = 0$ Hubbard model on a square lattice, we have $dD/dn > 0$ for $n < 1$ as shown by the solid curve in Fig. 1, while $dD/dn < 0$ for $n > 1$. Thus for $n > 1$ the total current may be well

regarded as being carried by holes in the filled band. On the other side, in one dimensional systems, D vanishes continuously not only for $n \to 0$ and $n \to 2$ but also for $n \to 1$ for finite interaction $U > 0$.[5] Therefore, in this case, even in the underdoped regime $n \lesssim 1$ the current-carrying carrier is regarded as holes doped in an insulating state at $n = 1$. From this point of view, it is interesting to note that our result for n/m' ($\propto D$) shows that dD/dn for $U/4t = 1$ first decreases from a positive value as n increases from 0, and finally it becomes negative for $n \lesssim 1$. The result indicates a hole-like behavior around half filling, $n \lesssim 1$. This behavior is caused by the fact that the slope dD/dn for $U = 0$ vanishes as $n \to 1$, while a finite negative correction $(n/m')_2$ decreases in this limit, as Umklapp processes becomes effective. When the Umklapp processes are effective, the Drude weight is reduced by the many-body effect, so that a large coupling constant $\rho^* U$ gives rise to a large difference $(n/m')_0 - (n/m')_2$, where ρ^* is the density of states at the Fermi level. Thus the decreasing derivative dD/dn for $n \lesssim 1$ as a function of U (> 0) is generally expected even in the weak coupling regime. For example, in our model, we assumed the coupling constant $U/W = 0.5$ for the bandwidth $W = 8t$. It is noted however that our results do not imply $D \propto |1 - n|$ for $n \to 1$ as in one dimensional systems, and that the hole-like behavior determined from the sign of dD/dn has nothing to do with the shape of the Fermi surface, namely, whether it is closed around $\boldsymbol{k} = (\pi, \pi)$. In effect, owing to the complete-nesting property occurring at $n = 1$, we should have to take account of antiferromagnetic ordering which sets in around half filling. Our results shown in Fig. 1 are valid when the ground state is a paramagnetic Fermi liquid.

In a strong coupling regime, the decrease in the Drude weight around half filling is reproduced by a numerical technique.[17] In our example of the weak coupling theory, however, the steep decrease in $(n/m')_2$ of eq. (4.18) is mainly caused by eq. (4.15), as is clear from the figure. In other words, the behavior $D \to 0$ as $n \to 1$ is ascribed to Umklapp processes becoming effective in this limit, rather than the decrease of the total weight n/m, eq. (3.49). Therefore, in regard to the mechanism to reduce D, our result should not be simply compared with the situation in the strong coupling regime, as in the t-J model,[18] where the decreasing behavior of D would be mainly due to $0 \le D/\pi e^2 = n/m' \le n/m \to 0$.

§5. Discussions

From the above results, it is remarked that the large quasiparticle mass m^* does not necessarily imply the enhanced mass m' in the Drude weight. This point is evident from the fact that, in an isotropic system, the former is affected by the many-body effect while the latter is not. Physically this means that a small Drude weight does not necessarily suggest a large specific heat coefficient. Thus, although m' and m^* are related with each other (eqs. (2.13) and (2.14)), they are practically regarded as independent quantities. In particular, this should be the case when we know little about the function $f(k, k')$ by the first principle calculation. We introduced the mass m' for the

sake of convenience so that it represents the mass per particle of the system as a whole,

$$\frac{E_p}{N} = \frac{p^2}{2m'}.$$

This is obtained from eq. (3.61). In this form it may become even clear why electron-electron interactions do not modify m' in a Galilean invariant system; where the electron-electron interaction cannot change the inertial mass of the system as a whole. The effective mass m' thus must be positive for the system to be stable.[19] As is clear from eq. (2.13), to distinguish m' from m^*, we must distinguish between the current and velocity of quasiparticle, and thereby we should not neglect the vertex correction. The difference is due to the Fermi liquid effect.

If a Fermi liquid state were continuously driven to the metal-insulator transition, we should have $m' \to \infty$. To see how this occurs, from eqs. (2.6) and (2.14) we may write

$$\frac{1}{m'} = \frac{1}{m^*}\left(1 + \frac{F_1^s}{3}\right), \tag{5.1}$$

where

$$\frac{n}{m^*} \equiv \frac{1}{3}\sum_k \boldsymbol{v}_k \cdot \boldsymbol{v}_k \delta(\mu - \varepsilon_k), \tag{5.2}$$

$$\frac{F_1^s}{3} \equiv \frac{\sum_{k,k'} f(k,k')\boldsymbol{v}_k \cdot \boldsymbol{v}_{k'}\delta(\mu - \varepsilon_k)\delta(\mu - \varepsilon_{k'})}{\sum_k \boldsymbol{v}_k \cdot \boldsymbol{v}_k \delta(\mu - \varepsilon_k)}. \tag{5.3}$$

Thus, to obtain $m' \to \infty$, we must have either (i) $m^* \to \infty$ or (ii) $F_1^s \to -3$. The former, if the coupling F_1^s remains unaffected, corresponds to reduction of energy scale of the system, with no qualitative change of the low-energy physics. In this case, the total weight n/m, eq. (3.49), will be reduced as well. The latter, usually unnoticed, is a nontrivial possibility, which reminds us of the ferromagnetic instability $F_0^a \to -1$ in ^{3}He. In this case, the degeneracy temperature below which to validate the Fermi liquid theory will be severely suppressed as we approach the transition point $D = 0$. We must be in this situation to conclude $D \to 0$ for finite $n/m \ (> 0)$, and this was the case of our concern in the previous section. Nonetheless, in either case, the instability does not manifest itself as a thermodynamic phenomenon, for the vanishing of the Drude weight makes sense only in a coherent regime. Although the case (ii) and its physical consequences are interesting in its own right as a way to destabilize the Fermi liquid state, here we do not discuss this point any further but to point out the possible instability. In general, (i) and (ii) is to be regarded as independent possibilities, although these are related with each other in an isotropic system, for which $m' = m < \infty$ because of the Landau relation $(1 + F_1^s/3)/m^* = 1/m$. In a lattice system, on the other side, the relation does not hold so that in principle we can even think of the hypothetical system where the vertex correction due to F_1^s is neglected while the mass m^* is heavily enhanced. In effect, the effective mass m' is generally anisotropic and the instability $m' \to \infty$ may be relevant particularly in an anisotropic system, *e.g.*, in a quasi-two-dimensional system where Umklapp processes can become quite effective in the direction perpendicular to a two-dimensional

layer. Then, beyond the instability, if any, the Fermi surface has to be rearranged in the way the system avoids the instability.

Finally we shall note that the plasma mode softens as $\omega_p \propto 1/\sqrt{m'}$ when $m' \to \infty$: In a charged system, the dielectric function $\epsilon(\omega)$ is related to the dynamical conductivity,

$$\epsilon(\omega) = 1 + \frac{4\pi\mathrm{i}}{\omega}\sigma(\omega). \tag{5.4}$$

Therefore, eq. (2.5) gives

$$\epsilon(\omega) = 1 - \frac{4\pi n e^2}{m'\omega^2},$$

from which the plasma frequency is given by

$$\omega_p^2 = \frac{4\pi n e^2}{m'}. \tag{5.5}$$

This result is valid only for $\omega_p \ll \varepsilon_{k_f}^*$, since we used the Fermi liquid formula for $\sigma(\omega)$.

In summary, we derived a formula to calculate the Drude weight D on the basis of the Fermi liquid theory. To this end, we considered the state $|p\rangle$ which is obtained from the ground state by boosting it by p in the momentum space. Thereby we identified the instability of the Fermi liquid state caused by the metal-insulator 'transition' $D \to 0$. In a lattice system, electron-electron interactions enhance the effective mass m', although an unrenormalized mass $m' = m$ is concluded identically in a Galilean invariant system. As the enhancement is caused by Umklapp processes, the many-body effect on m' is largest in the vicinity of half filling $n \sim 1$ in a single-band model. This was shown for the Hubbard model in a square lattice by perturbation theory. As a result, even in a weak-coupling regime, we obtained the Drude weight D showing the behavior of the 'doped insulator' $\mathrm{d}D/\mathrm{d}n < 0$ in the underdoped region $n \lesssim 1$. It was noted that m' corresponds to the mass in the quasiparticle current j_k and is generally different from the thermal mass m^* defined through the velocity of quasiparticle v_k^*.

Acknowledgments

Appendix: Finite Temperature Formalism

In this appendix, we outline the derivation of eq. (2.6) by the finite temperature diagram technique. The formalism used here is based on the work of Éliashberg.[20] According to eq. (24) of

ref. 20, the conductivity is given by

$$\sigma_{\mu\nu}(\omega) = \frac{ie^2}{2\Omega} \left\{ \sum_k v^*_{k\mu} \frac{1}{2T} \frac{\cosh^{-2}(\varepsilon^*_k - \mu/2T)}{\omega + 2i\gamma^*_k} v^*_{k'\nu} + \frac{1}{2} \sum_{k,k'} z_k v^*_{k\mu} \frac{1}{2T} \frac{\cosh^{-2}(\varepsilon^*_k - \mu/2T)T_{22}(k,k';\omega)}{(\omega + 2i\gamma^*_k)(\omega + 2i\gamma^*_{k'})} z_{k'} v^*_{k'\nu} \right\},$$

(A·1)

where

$$\gamma^*_k = -z_k \mathrm{Im}\,\Sigma(k,\omega)|_{\omega=0}.$$ (A·2)

For our purpose we shall use only the real part of $T_{22}(k,k';\omega)$, for which eq. (12) of ref. 20 gives

$$\mathrm{Re}T_{22}(k,k';\omega) = \left(\tanh \frac{\varepsilon^*_{k'} - \mu + \omega}{2T} - \tanh \frac{\varepsilon^*_{k'} - \mu}{2T} \right) \mathrm{Re}\Gamma(k,k';\omega).$$ (A·3)

Therefore, noting

$$\frac{1}{4T} \cosh^{-2} \frac{\varepsilon^*_k - \mu}{2T} \to \delta(\mu - \varepsilon^*_k), \qquad (T \to 0)$$

and $\gamma^*_k \propto T^2$, in the collisionless regime $\gamma^*_k/\omega \to 0$ of our concern, the limit $\omega \to 0$ leads to

$$\sigma_{\mu\nu}(\omega) \to \frac{ie^2}{\omega} \frac{1}{\Omega} \sum_k \left\{ v^*_{k\mu} v^*_{k\nu} \delta(\mu - \varepsilon^*_k) + \sum_{k'} f(k,k') v^*_{k\mu} v^*_{k'\nu} \delta(\mu - \varepsilon^*_k) \delta(\mu - \varepsilon^*_{k'}) \right\}$$ (A·4)

$$= \frac{ie^2}{\omega} \frac{1}{\Omega} \sum_k v^*_{k\mu} j^*_{k\nu} \delta(\mu - \varepsilon^*_k),$$ (A·5)

where $f(k,k')$ is defined in eq. (3.42).

In the opposite hydrodynamic limit, $\omega/\gamma^*_k \to 0$, the imaginary part of $T_{22}(k,k';\omega)$ plays an important role, and in contrast to the real part, to investigate $\mathrm{Im}T_{22}(k,k';\omega)$ poses a problem. This rather complicated task is indispensable, e.g., to evaluate the dc conductivity $\sigma(0)$ ($\propto T^{-2}$) of a clean system at finite temperature.[21] This point will be discussed in detail in the following paper.[7]

[1] W. Kohn: Phys. Rev. **133** (1964) A171.

[2] B. S. Shastry and B. Sutherland: Phys. Rev. Lett. **65** (1990) 243.

[3] V. J. Emery, A. Luther and I. Peschel: Phys. Rev. **13** (1976) 1272.

[4] T. Usuki, N. Kawakami and A. Okiji: Phys. Lett. **135A** (1989) 476.

[5] H. J. Shultz, in *Correlated Electron Systems,* edited by V. J. Emery (World Scientific, Singapore, 1992).

[6] K. Kanki and K. Yamada: J. Phys. Soc. Jpn. **66** (1997) 1103.

[7] T. Okabe: J. Phys. Soc. Jpn. (submitted).

[8] D. Pines and P. Nozières: *The Theory of Quantum Liquids I* (W. A. Benjamin, Inc., New York, 1966).

[9] The state $|p\rangle$ will be stable because a finite lifetime of the state $|p\rangle$ implies a finite dc conductivity at $T = 0$.

[10] In a homogeneous system, the relation between $\partial^2 E_p/\partial p^2$ and the Drude weight reminds us of the derivation of the inequalities $F^s_l/(2l + 1) + 1 > 0$ by investigating the stability of the Fermi liquid state against an infinitesimal change in shape of the Fermi surface, $\delta n^s = \sum_{lm} \delta n^s_{lm} Y_{lm}(\theta,\phi)$.[8] Here δn^s, a spin symmetric part of a deviation from the ground-state distribution n^0_k, is expanded in terms of spherical harmonics $Y_{lm}(\theta,\phi)$ for the polar coordinates (θ,ϕ) on the Fermi surface. However, in this argument, the case $l = 1$ is an exception when the parameter F^s_1 is canceled out in the increase of the total energy δE^s_l caused by the deformation, i.e.,

$\delta E_l^s \propto (1+F_l^s/3)/m^* \propto (1+F_l^s/3)/(1+F_1^s/3)$. The cancellation ensures the stability of the Fermi surface and an unrenormalized Drude weight. The condition $1+F_1^s/3 > 0$ is still required for the effective mass $m^* = (1+F_1^s/3)m$ to be positive definite.

[11] It is noted that the distribution function of noninteracting particles coincides with the distribution function of quasiparticles in an interacting system.

[12] P. Nozières: *Theory of Interacting Fermi Systems* (Benjamin, New York, 1964).

[13] A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinskii: *Methods of Quantum Field Theory in Statistical Physics* (Pergamon, Oxford, 1965).

[14] Nevertheless, the part of $\lim_{\omega \to 0} \sigma_{\mathrm{inc}}(\omega)$ due to quasiparticle scattering is obtained by taking into account the effect of the quasiparticle damping.[7]

[15] L. D. Landau and E. M. Lifshitz: *Quantum Mechanics* (Pergamon, Oxford, 1977).

[16] H. Maebashi and H. Fukuyama: J. Phys. Soc. Jpn. **66** (1997) 3577.

[17] See, E. Dagotto, A. Moreo, F. Ortolani, D. Poilblanc and J. Riera: Phys. Rev. **45** (1992) 10741, and references therein.

[18] See, for example, A. J. Millis and S. N. Coppersmith: Phys. Rev. **42** (1990) 10807; J. Igarashi and P. Fulde: Phys. Rev. **48** (1993) 12713.

[19] A negative D was observed and discussed in the numerical studies of finite systems; J. Wagner, W. Hanke and D. J. Scalapino: Phys. Rev. **43** (1991) 10517; C. A. Stafford, A. J. Millis and B. S. Shastry: Phys. Rev. **43** (1991) 13660.

[20] G. M. Éliashberg: Sov. Phys. JETP **14** (1962) 886.

[21] K. Yamada and K. Yosida: Prog. Theor. Phys. **76** (1986) 621.